Formulas of the Perspective Cartographic Projection for Planets and Asteroids of Arbitrary Shape

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Abstract

Formulas of transformation between coordinates on image plane, planetocentric coordinates and photometric conditions of observation for arbitrary planet have been obtained. An example with ellipsoidal planet has been considered.

1 Introduction

When processing planetary images it is necessary to transform image coordinates to planetocentric coordinates or back. Also at photometric studies it is necessary to calculate photometric conditions (geometry) of observation for each point on the planet surface.

Often at realization of such transformations it is supposed that the planet image is in orthographic projection. But it is true only if ratio of the planet size to distance to it (*i.e.*, the angular size of the planet) is negligible for accuracy of the task being solved. Otherwise it is necessary to suppose that the planet image is in the perspective projection (this is especially appreciable at the observations from the board of space vehicles approaching with planets and asteroids).

Formulas of the perspective projection are often obtained for a spherical planet or for some special cases of non-spherical planets. We have tried to make a step to deriving formulas of coordinate transformation in most general form: let us consider images of planets of arbitrary shape in the perspective azimuthal projection. Let us suppose that it is possible to set the form of a planet with equation $F(x, y, z) = 0$ or a set of such equations for different parts of the surface. Let us suppose also, that the line of sight is not directed strongly to the center of the planet, *i.e.*, we deal with more general Tilted Perspective Projection rather than Vertical Perspective Projection.

2 Problem definition

Let us choose such a system of rectangular coordinates (XYZ) that it is most suitable to set the shape of the planet surface with an equation:

$$F(x, y, z) = 0. \tag{1}$$

If the planet is an ellipsoid, equation (1) can be written like this:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1, \tag{2}$$

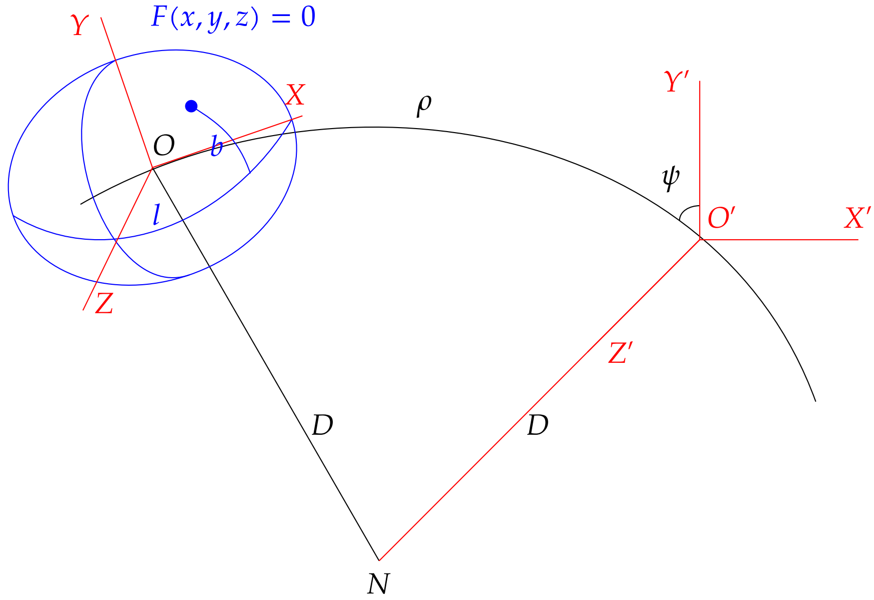

Figure 1: Coordinate systems

where A, B, and C — ellipsoid semi-axes. Let us name the coordinate origin (point O) of system (XYZ) "the center of the planet" (figure 1). Let us choose a plane of the perspective projection such that the line of sight crosses it at right angles in a point, placed on the distance D from the observer, where D — distance between the observer and the center of the planet. In this plane we shall consider the image (projection) of a planet, that is equivalent to the image on a photodetector (as a rule, it is a rectangular CCD-image). Let us set coordinates on the image (on the projection plane) x_P and y_P (let axis X_P be directed to the right, and Y_P — upward). Let us introduce an additional rectangular system of coordinates ($X'Y'Z'$), with coordinate origin (point O') in the point of crossing of the line of sight with the projection plane (here axis Z' is directed to the observer, and axes X' and Y' coincide with axes X_P and Y_P correspondingly). In this system the observer has coordinates $x' = y' = 0$, $z' = D$. Let us note that at $D \to \infty$ the perspective projection approaches to orthographic one. Let coordinates in all systems be measured in the same units — for example, in image pixels. Converting to other units (to kilometers, angular seconds) can be performed, knowing scale of the image and distance to the planet. Since we use the Tilted Perspective Projection, let us consider, that the center of the planet is displaced relative to the direction of the line of sight on an angle ρ (on the coelosphere) in a direction with azimuth ψ counted in the image plane from the positive direction of axis Y' anticlockwise. If Vertical Perspective is enough (it is acceptable if the distance to the planet much more than its size), in all formulas it is possible to set $\rho = \psi = 0$. Let us consider the system of planetocentric coordinates: b — a latitude, l — a longitude. Let us define it as spherical system of coordinates with center in the center of the planet, the equator of which is placed in plane XOZ, axis Y is directed to a north pole, and axis Z is directed to the point with coordinates $b = l = 0$. And now let us derive the formulas of transformation from coordinates on the image plane (x_P, y_P) to planetocentric coordinates (b, l), and also inverse transformation and formulas of calculation of the observational photometric conditions.

3 Transformation $(x_P, y_P) \to (b, l)$

Coordinates on the CCD-image (x_P, y_P) are given.

3.1 Step A: $(x_P, y_P) \to (x, y, z)$

Let us write the equation of the right line corresponding to the line of sight (in system (XYZ)):

$$\frac{x - x_N}{x_A - x_N} = \frac{y - y_N}{y_A - y_N} = \frac{z - z_N}{z_A - z_N} \tag{3}$$

where (x_N, y_N, z_N) — coordinates of the observer, (x_A, y_A, z_A) — coordinates of a point on the projection plane corresponding to (x_P, y_P). Coordinates of the observer can be obtained like this:

$$\begin{cases} x_N = D \sin l_0 \cos b_0 \\ y_N = D \sin b_0 \\ z_N = D \cos l_0 \cos b_0 \end{cases}, \tag{4}$$

where b_0, l_0 — planetocentric coordinates of the point under the observer. Coordinates (x_A, y_A, z_A) can be obtained from (x_P, y_P) by transforming from $(X'Y'Z')$ to (XYZ) with several consecutive rotations and translations of axis. Let $(x', y', z') = (x_P, y_P, 0)$, and make this transformation (if $\rho = \psi = 0$ then step 1 can be omitted):

1. transformation to system of coordinates (x_1, y_1, z_1) with coordinate origin in the center of planet:

   (a) rotation on angle ψ around axis Z' to place the center of planet in plane $Y_{1a}O'Z_{1a}$:

   $$\begin{cases} x_{1a} = x' \cos \psi + y' \sin \psi \\ y_{1a} = -x' \sin \psi + y' \cos \psi \\ z_{1a} = z' \end{cases};$$

   (b) translation of coordinate origin to the observer point N:

   $$\begin{cases} x_{1b} = x_{1a} \\ y_{1b} = y_{1a} \\ z_{1b} = z_{1a} - D \end{cases};$$

   (c) rotation on angle ρ around axis X_{1b} to place the center of planet on axis Z_{1c}:

   $$\begin{cases} x_{1c} = x_{1b} \\ y_{1c} = y_{1b} \cos \rho + z_{1b} \sin \rho \\ z_{1c} = -y_{1b} \sin \rho + z_{1b} \cos \rho \end{cases};$$

   (d) translation of coordinate origin to the center of planet:

   $$\begin{cases} x_1 = x_{1c} \\ y_1 = y_{1c} \\ z_1 = z_{1c} + D \end{cases}.$$

2. rotation on position angle to place a central meridian of the planet along axis Y_2:

   $$\begin{cases} x_2 = x_1 \cos(P_0 - \psi) + y_1 \sin(P_0 - \psi) \\ y_2 = -x_1 \sin(P_0 - \psi) + y_1 \cos(P_0 - \psi) \\ z_2 = z_1 \end{cases},$$

   where P_0 — the position angle of the planet on the image, counted from position "the north is above" anticlockwise (let us note that if not $\rho = \psi = 0$, planes X_1OY_1 and $X'O'Y'$ are not parallel, therefore a such "position angle" should be separated in two parts: ψ (in X_1OY_1) and $P_0 - \psi$ (in $X'O'Y'$), where $P_0 - \psi$ is position angle, counted from direction from the center of planet to the center of Tilted Perspective Projection).

3. rotation on angles b_0 and l_0 to transform to system (XYZ):

$$\begin{cases} x_3 = x_2 \\ y_3 = y_2 \cos b_0 + z_2 \sin b_0 \\ z_3 = -y_2 \sin b_0 + z_2 \cos b_0 \end{cases}$$

$$\begin{cases} x = x_3 \cos l_0 + y_3 \sin l_0 \\ y = y_3 \\ z = -x_3 \sin l_0 + z_3 \cos l_0 \end{cases}$$

This is a general form of transformation $(X'Y'Z')$ to (XYZ). In our case $(x_A, y_A, z_A) = (x, y, z)$. Or in the matrix presentation this transformation looks like this:

$$\begin{bmatrix} x_A \\ y_A \\ z_A \\ 1 \end{bmatrix} = \begin{bmatrix} \cos l + 0 & 0 & \sin l_0 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin l_0 & 0 & \cos l_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos b_0 & \sin b_0 & 0 \\ 0 & -\sin b_0 & \cos b_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} \cos(P_0 - \psi) & \sin(P_0 - \psi) & 0 & 0 \\ -\sin(P_0 - \psi) & \cos(P_0 - \psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & D \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \quad (5)$$
$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\rho & \sin\rho & 0 \\ 0 & -\sin\rho & \cos\rho & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -D \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\psi & \sin\psi & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_P \\ y_P \\ 0 \\ 1 \end{bmatrix}$$

To calculate coordinates of a point on the surface of the planet, we should solve system (3) together with equation (1), finding unknown x, y, z. We should (if it is necessary) find all solutions — *i.e.*, all points of crossing of the right line with the surface and to choose a point with smallest distance to the observer, *i.e.*, with smallest value of $(x_N - x)^2 + (y_N - y)^2 + (z_N - z)^2$.

In the case of ellipsoid (equation (2)) the problem is reduced to solving the quadratic equation. This solution can be written as follows:

$$\begin{cases} z = \dfrac{1}{c_2}\left(-c_1 \pm \sqrt{c_1^2 - c_2(a_2^2 + a_4^2 - 1)}\right) \\ y = A(a_1 z + a_2) \\ z = B(a_3 z + a_4) \end{cases} \quad (6)$$

where

$$a_1 = \frac{1}{A} \cdot \frac{x_A - x_N}{z_A - z_N} \quad a_2 = \frac{x_N}{A} - a_1 z_N$$
$$a_3 = \frac{1}{B} \cdot \frac{y_A - y_N}{z_A - z_N} \quad a_4 = \frac{y_N}{B} - a_3 z_N$$
$$c_1 = a_1 a_2 + a_3 a_4 \quad c_2 = a_1^2 + a_3^2 + 1/C^2$$

The one of the two solutions of equation (6) with smaller $(x_N - x)^2 + (y_N - y)^2 + (z_N - z)^2$ corresponds to nearside of the planet.

3.2 Step B: $(x, y, z) \to (b, l)$

Under the (b, l) definition we have:

$$\begin{cases} b = \arcsin\left(y/\sqrt{x^2 + y^2 + z^2}\right) \\ l = \arctan(x/z) - \pi \, \mathrm{sign}(z)(1 - \mathrm{sign}(z))/2 \end{cases}$$

Here the term with sign functions removes ambiguity of arctan function.

4 Transformation $(b, l) \to (x_P, y_P)$

Planetocentric coordinates (b, l) are given.

4.1 Step A: $(b, l) \to (x, y, z)$

Coordinates of a point on the planet surface (x, y, z) can be found with solving system of the equations:

$$\begin{cases} x\cos l - z\sin l = 0 \\ y\cos b - (x\sin l + z\cos l)\sin b = 0 \\ F(x, y, z) = 0 \end{cases} \tag{7}$$

where the first two equations describe a right line corresponding to direction from the center of planet with spherical coordinates (b, l); and the last equation describe surface of the planet; and we find the cross point.

In the case of ellipsoid the solution looks like this:

$$\begin{cases} x = r(b, l)\cos b\sin l \\ y = r(b, l)\sin b \\ z = r(b, l)\cos b\cos l \end{cases}$$

where $r(b, l) = 1 \Big/ \sqrt{\frac{\cos^2 b\sin^2 l}{A^2} + \frac{\sin^2 b}{B^2} + \frac{\cos^2 b\cos^2 l}{C^2}}$.

4.2 Step B: $(x, y, z) \to (x_P, y_P)$

Let us convert coordinates (x, y, z) to system $(X'Y'Z')$ with transformation that is inverse to equation (5):

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & D \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\rho & -\sin\rho & 0 \\ 0 & \sin\rho & \cos\rho & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -D \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(P_0 - \psi) & -\sin(P_0 - \psi) & 0 & 0 \\ \sin(P_0 - \psi) & \cos(P_0 - \psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos b_0 & -\sin b_0 & 0 \\ 0 & \sin b_0 & \cos b_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos l + 0 & 0 & -\sin l_0 & 0 \\ 0 & 1 & 0 & 0 \\ \sin l_0 & 0 & \cos l_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The point (x', y', z'), observer point $(0, 0, -D)$ and point $(x_P, y_P, 0)$ (in system $(X'Y'Z')$) are placed on one right line. Thus following relation has to be valid:

$$\frac{x_P}{x'} = \frac{y_P}{y'} = \frac{-D}{z' - D}$$

Therefore $x_P = \frac{x'D}{D - z'}$, $y_P = \frac{y'D}{D - z'}$

5 Transformation $(x, y, z) \to (\alpha, i, \varepsilon)$

To find the phase angle α, the angles of incidence i and emergence ε, let us use coordinates (x, y, z), obtained in the Step A in any of described above transformations.

Let us obtain a direction of a normal $\vec{n}$ to the surface, differentiating function equation (1) in point (x, y, z):

$$\vec{n} = \left(\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz} \right)$$

and then obtain a vector of direction to the observer $\vec{N}$:

$$\vec{N} = (x_N - x, y_N - y, z_N - z)$$

where (x_N, y_N, z_N) — coordinates of the observer from equation (4). The angle of emergence ε we find as the angle between vectors $\vec{n}$ and $\vec{N}$:

$$\cos \varepsilon = \left| \frac{(\vec{n}, \vec{N})}{|\vec{n}| \cdot |\vec{N}|} \right|$$

To obtain the angle of incidence i and the phase angle α we should calculate coordinates of the Sun. We know planetocentric coordinates of the point on the surface under the Sun (b_S, l_S) and distance from the Sun to the planet center r_S (as a rule, it is possible to set r_S equal to infinity). Let us obtain Sun coordinates in system (XYZ):

$$\begin{aligned} x_S &= r_S \sin l_S \cos b_S \\ y_S &= r_S \sin b_S \qquad . \\ z_S &= r_S \cos l_S \cos b_S \end{aligned}$$

Thus the vector of a direction to the Sun is

$$\vec{S} = (x_S - x, y_S - y, z_S - z)$$

Then the angle of incidence i and phase angle α can be calculated like this:

$$\cos i = \left| \frac{(\vec{n}, \vec{S})}{|\vec{n}| \cdot |\vec{S}|} \right| \qquad \cos \alpha = \left| \frac{(\vec{S}, \vec{N})}{|\vec{S}| \cdot |\vec{N}|} \right|$$

Often instead of values (α, i, ε) it is convenient to use values (α, j, l), where j — photometric latitude, l — photometric longitude. By definition:

$$\begin{aligned} \cos i &= \cos j \cos(\alpha - l) \\ \cos \varepsilon &= \cos j \cos l. \end{aligned}$$

6 Conclusion

Thus, using above mentioned transformations of coordinates, it is possible to process images of any objects, shape of which can be presented like equation (1). The observer and a light source can be placed on arbitrary distance from the planet. We should use the following constants, determining observational conditions: $b_0, l_0, P_0, D, b_S, l_S, r_S, r, y$. If it is impossible to set the shape of a planet analytically it is necessary to use its discrete representation and then the solution of systems equation (3)+(1) and equation (7) has to be found numerically.

Appendixes

A Tangential distortion in planetography library

A.1 Definitions

Definition 1. *Tangential distortion is the dependence of angular scale on the distance from the optical axis in the ideal camera.*

In the ideal camera (figure 2) all beams go through the optical center of the camera. If registration surface is a plane, then angle Δs between directions to the optical center O from two close points (P_R^1 and P_R^2) in the registration plane decreases with increase of the distance l of the points from the projection of optical center to the registration plane (O_R). However, in the projection

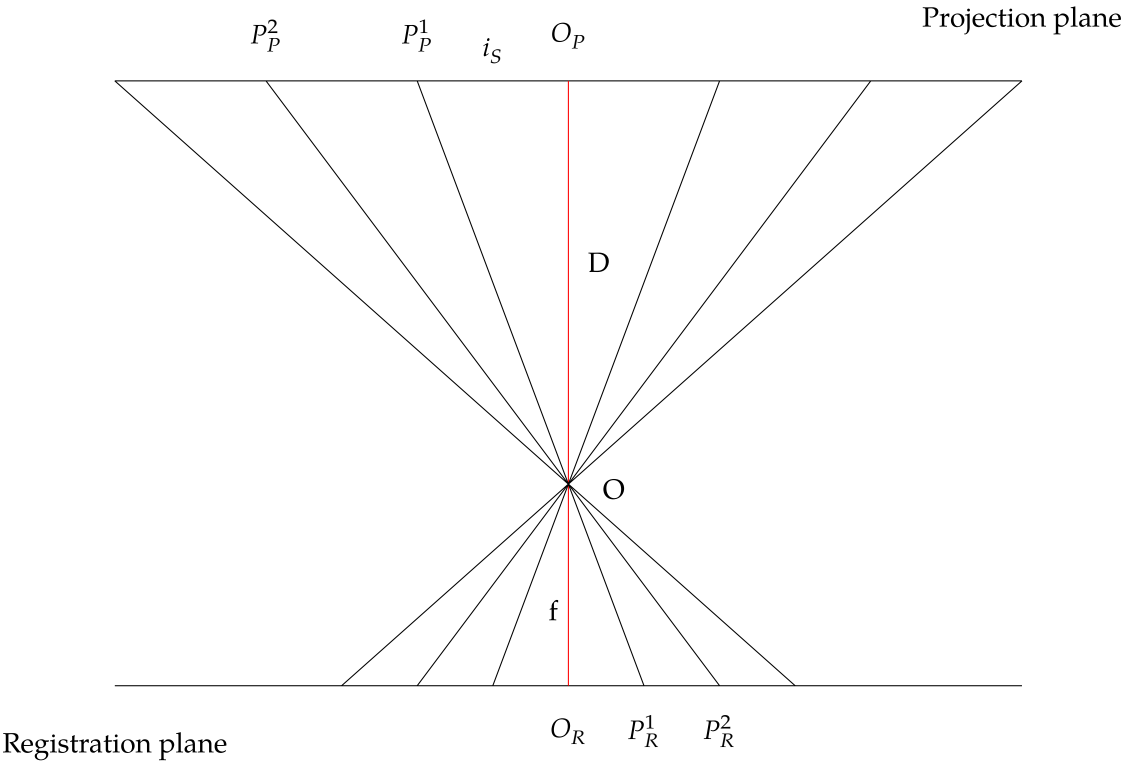

Figure 2: Ideal camera

plane the distance between corresponding points does not depend on the distance from optical center projected to the projection plane (O_P).

A.2 Planetography implementation

Transformation from a camera image (which is in tilted perspective projection in general case) is based on calculations of lines of sight for center of each pixel and their intersection with the object body (we know the equation of straight line for the line of sight and equation of the object shape. And we can solve this equation system to find the intersection point(s)). For calculations of line of sight for a particular pixel x, y (*i.e.*, the straight line in 3-dimensional space via camera position and the center of pixel in the projection plane), the scales of image in linear units iS_x, iS_y are used. This scale is the size of a pixel in the projection plane. Since this size is the same for every pixel, one can calculate it for the central one via tan of the $iFoV$ and distance to the object $iS = \tan(iFoV) \cdot D$, where D is the distance to the projection plane.